Resum de les conferències convidades

Integral geometry using spheres

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Integral geometry in Euclidean space uses the affine subspaces, slicing a submanifold, counting contacts, and eventually averaging on a Grassmann manifold, which is compact. Many results obtained during the last century in integral geometry are of the type: topology implies geometry. A paradigm is the theorem of Fary, Fenchel and Milnor: the lower bound of the total curvature of a knotted curve is 4π, that is at least 2π larger than the value for a circle.

Conformal geometry in R^n or S^n needs to use a family of submanifolds invariant by the group of global conformal transformations. The simplest are the spheres. This means we need to provide the set of spheres with an invariant measure.

We will explain a result in S^3 which is a conformal analogous to the theorem of Fary, Fenchel and Milnor. A sphere can intersect a circle at most in two points (if it does not contain the circle). To measure "how many" spheres intersect a closed curve in four points or more tells how far it is from a circle. Our theorem is: if the curve is knotted, this measure cannot be too small. This measure is not equal to the integral of a geometric function defined on the curve, but it is the integral on the set of pairs of points on the curve of a "bilocal" function. This result, contained in a joined paper with J. O'Hara, is obtained using the modulus of zones between two spheres in special position with respect to the knot. A similar statement about non-separable links, obtained with G. Moniot, will be explained.

Stereology: a science between mathematical play and practical need

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When the *International Society for Stereology* was founded in 1961, the term 'stereology' (from the Greek 'στερεοζ' = solid) was coined to mean 'extrapolation from 2- to 3-dimensional space'. More generally, stereology may be regarded as inference of geometric properties of spatial structures from properly sampled information or, in short, as 'geometric sampling'.

Few mathematicians have enjoyed the privilege of witnessing the implementation of their abstract ideas in practice: Luis Santaló was one of them. In his joint book with J. Rey Pastor [*Geometria Integral*, Buenos Aires, 1951], Eq.(32.7) reads (in the original notation):

$$\int n\, dG = \pi F ,$$

where dG is the element of invariant measure for straight lines in R^3, and n the number of intersections between a straight line G and a fixed surface S of area F; the integral is extended to the set of straight lines which hit the surface. Likewise,

$$\int l\, dG = 2\pi V ,$$

where l is the intercept length determined by G in a bounded subset C of volume V. A probabilist would construct an element of probability measure $dG / \int dG$ for an invariant straight line G hitting C. If $S \subset C$, then the preceding two equations imply

$$\frac{F}{V} = 2\frac{E(n)}{E(l)} ,$$

which is one of the classical equations of stereology; $E(.)$ denotes mean value or expectation with respect to the mentioned probability measure. Further, a statistician would design a experiment in which a finite number of invariant straight lines are used as geometric probes to sample C with the idea of estimating the target parameter F. Assuming that V was known, the natural estimator of F would be the ratio estimator

$$\hat{F} = V.2\frac{\sum n}{\sum l} ,$$

where Σn and Σl are the total observed counts and intercept lengths, respectively, actually measured in the experiment. Finally, a practical stereologist could apply the preceding result to estimate for instance the total surface area of alveoli in a lung by visualizing an isotropic uniform random test line in three dimensions 'piercing' the alveolar surface. The practical implementation starts by cutting the lung exhaustively into slices of approximately equal thickness. A systematic sample of slices is laid flat on a table, and a uniform systematic subsample of small tissue blocks is obtained (at the vertices of a uniformly overlaid square lattice, for instance). Each block is suitably processed and a thin section is cut through it with an isotropic random orientation (to comply with the integral geometric dictate). Finally, under the microscope a test system of line segment probes of known properties is superimposed on the section and the relevant intersections between the test lines and the alveolar surface (which appears as a curve trace in the section) are counted. In this way one can say that the alveolar surface area of the lung of a 70 kg human adult is about 130 m^2 (close to the area of a tennis court). Ewald R. Weibel pioneered the application of stereology to lung [*Morphometry of the Human Lung*, New York 1963]; the author was one of his coworkers at the University of Bern between 1976 and 1994. Other relevant quantities, seemingly inaccessible *a priori*, can be estimated with a moderate amount of effort -for instance the number of neurons in the cortex of a human male brain is about 25×10^9 (with important variation among individuals). Some quantities (such as organ, muscle, or fat volumes) can nowadays be estimated *in vivo* with non invasive scanning technologies such as magnetic resonance imaging.

The relevant integral geometric tools for stereology evolved from unbounded and independent affine space probes into bounded and systematic ones, which are easier to apply. A fundamental mathematical ingredient for geometric systematic sampling is an integral geometric formula for lattices [*Integral Geometry and Geometric Probability*, 1976, Ch.8] with which L. Santaló was playing from very early years, see for instance his Geometria Integral 31, *Hamburg. Abh*., 1940. (This paper has also a sentimental value: it was signed in Rennes in June 1939, when Santaló was a refugee from the Spanish civil war).

Historically, however, practical stereology evolved independently of the progress of integral geometry, and *vice versa*. For instance, the French geologist August Delesse proposed in 1847 to estimate the volume proportion of a mineral in a rock by the corresponding area ratio in a section. Thereafter, in 1933 the Russian geologist A. A. Galgolev proposed to estimate section areas by point counting with a test system of points, which is an implicit application of Santaló's result. The real connections began to be properly formalized only in the 1970's thanks to the influential work of Roger E. Miles.

Recently a number of stereological tools have been developed which are based on linear rather than affine subspaces, acted upon by the group of rotations, e.g. test lines through a fixed point, test planes though a fixed line, etc. [For general reference see Eva B. Vedel Jensen (1998) *Local Stereology*]. Another line of current interest is the prediction of the precision of systematic sampling with geometric probes on Euclidean spaces, on the circle, and on the sphere.

El pensamiento de Lluís A. Santaló, guía ante los problemas de nuestra educación matemática

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Lluís Santaló fue un matemático muy sobresaliente, por supuesto. Pero, probablemente por encima de eso, en su propia valoración, fue un maestro. Y por cierto, un gran maestro. Una de las tareas a las que dedicó con gran eficacia una buena parte de su energía consistió en pensar a fondo sobre los problemas de la enseñanza de la matemática. Él transmitió en numerosas obras, con la claridad y el buen sentido característicos de su pensamiento, sus diagnósticos sobre la situación de la educación matemática a nivel global y local. Y en su exploración no descuidó ninguno de los niveles de enseñanza. Santaló se expresó bien explícitamente y en numerosas ocasiones sobre las repercusiones que una enseñanza mal orientada o inadecuadamente realizada podría tener sobre la marcha misma del desarrollo cultural de un país e incluso sobre la calidad y cantidad de la investigación matemática en un ambiente determinado.

En esta conferencia, mi contribución al homenaje a Lluís Santaló, trataré de aplicar su pensamiento a algunos de los problemas concretos de la situación de la educación matemática en nuestro país.